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Transmission Properties of Random Point Scatterers for Waves with Two-Band Dispersion Law

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The scattering problem is considered for the one-dimensional Dirac equation whose potential is a system of randomly distributed point scatterers. Types of such scatterers are described. The probability densities for the transmission and transformation disbalance coefficients are determined in the high-energy region and various averaged characteristic are calculated.

KEY WORDS: Wave transmission; dispersion law; point scatterers; Dirac equation.

1. INTRODUCTION

There is an extensive literature describing the scattering problem for the Schrödinger equation with a random potential (see e.g., ref. 1, Chapter VII, for a review and references). This problem is, however, of great interest for the Dirac equation as well. Reasons to study it are the following: first, it is desirable to comprehend what complication or a new aspects may appear for a two-band spectrum, and second, there is a class of physical problems connected with the propagation of short, random impulses through regular media which reduce to the Dirac equation (written in a *moving* coordinate system) with a random point interaction potential. In particular, studying the nonlinear absorption of a stochastic acoustic signal by a superconductor, we arrive at the equation[2]

$$-iv(\sigma_z - \beta)\frac{d\psi}{dx} + \Delta\sigma_x\psi + \beta U(x)\psi = E\psi \tag{1}$$

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where $U(x)$ is a random potential, v is the particle velocity in the *laboratory* coordinate system, Δ is the dispersions law gap, s is the velocity of the reference frame connected with the acoustic wave (the sound velocity), $\beta = s/v$, and σ_x and σ_z are the Pauli matrices. Following ref. 3, we call (1) the tilted Dirac equation. Moreover, we say that $\beta < 1$ corresponds to the weakly tilted case and $\beta > 1$ to the strongly tilted case.

In one-dimensional disordered systems the scattering problem plays an important role. The corresponding scattering characteristics behave in a specific manner reflecting the well-known fact of state localization. For instance, the transmission coefficient for the Schrödinger equation turns out to be exponentially small as a function of the length of the disordered segment. Moreover, the scattering characteristics are directly related to some kinetic quantities of such systems. For instance, the electrical conductivity of a disordered segment of length L is given by the Landauer formula

$$G_L = \frac{e^2}{h} \frac{\langle T_i \rangle_{\mathrm{F}}}{\langle R_i \rangle_{\mathrm{F}}} \tag{2}$$

while its thermal conductivity is given by

$$K_L = \frac{\pi^2 k_{\mathrm{F}} T}{3h} \langle E^2 T_i \rangle \tag{3}$$

Here $\langle \cdot \rangle_{\mathrm{F,B}}$ denotes the averaging over energy with the weight $-\partial n_{\mathrm{F,B}}/\partial E$, where $n_{\mathrm{F,B}}$ is the Fermi (Bose) function and T_L and R_L are the transmission and reflection coefficients of the segment, respectively. (The derivation and discussion of these formulas can be found, for instance, in ref. 1.) The absorption rate of the sound by a superconductor is determined, analogously to (2) and (3), by the formula[2]

$$Q = \frac{1}{4} N(E_{\mathrm{F}})\, \beta^2 \int_{-\infty}^{\infty} \frac{d\beta'}{\beta'^3} \int_{-\infty}^{\infty} dE\, R_L(E)(\varepsilon_+ - \varepsilon_-)[n_{\mathrm{F}}(\varepsilon_+) - n_{\mathrm{F}}(\varepsilon_-)]$$

Here $N(E_{\mathrm{F}})$ is the density of states at the Fermi level and $\varepsilon_{\pm}(E)$ are the dispersion laws for the unperturbed problem [for $\beta U(x) = 0$] in the *laboratory* coordinate system.

The scattering and the corresponding spectral problem for Eq. (1) with a Markov-type potential have been considered in ref. 3. The main results obtained in ref. 3 are the following: in the weakly tilted case the mean transmission coefficient for a disordered barrier of length L is exponentially small for large L and all states of the infinite system are localized, i.e., its spectrum is pure point. In other words, in this case the structure of the solutions to the weakly tilted Dirac equation with a random potential qualitatively resembles the structure of solutions to the Schrödinger equation with a random potential (see, for instance, ref. 1). On the other hand, for the strongly tilted case the reflection is replaced by a partial transformation of waves between two scattering channels with a mean disbalance coefficient being exponentially small. For an infinite system all the states are delocalized and the spectrum is absolutely continuous.

One of the most popular models of the random potential is a potential generated by point scatterers randomly distributed over the axis. This potential serves as a basis for many exactly solvable models of one-dimensional disordered systems. In a recent paper,[(4)] for instance, the probability density of the transmission coefficients in the Schrödinger case was obtained explicitly for this potential in the high-energy limit.

In the present paper we study the properties of the tilted Dirac equation (1) with a potential $\beta U(x)$ generated by random point scatterers. As already mentioned, this problem arises in particular when investigating the transmission of random sequences of extremely short impulses through a regular medium and is interesting from the mathematical as well as from the physical point of view.[3] This is partially connected with the fact that the point potentials, which for the Dirac equation are more complicated than for the Schrödinger one, remain poorly understood. Particular attention is paid to the scattering problem. We compute the probability densities for the transmission and transformation coefficients in the high-energy limit in both the weakly and strongly tilted cases. It can be shown that the spectral properties of the problem on the whole line are fully analogous to those for the Markovian potential already discussed and we do not describe them here.

2. STATISTICAL PROPERTIES OF THE TRANSMISSION COEFFICIENT IN THE WEAKLY TILTED CASE[4]

In the weakly tilted case we write the solutions to Eq. (1) on the right and on the left sides of the disordered segment $(0, z_0)$ in the form

$$|\psi\rangle = \begin{cases} \psi_+(0)\,|u_+\rangle\, e^{ip_+ z} + \psi_-(0)\,|u_-\rangle\, e^{ip_- z}, & z<0 \\ \psi_+(z_0)\,|u_+\rangle\, e^{ip_+(z-z_0)} + \psi_-(z_0)\,|u_-\rangle\, e^{ip_-(z-z_0)}, & z>z_0 \end{cases}$$

[3] For more extensive physical motivation see ref. 2.

[4] Most of this section is a simplified version of ref. 4 adapted to the Dirac equation.

where $p_\pm(E)$ are the momenta of the plane wave solutions of Eq. (1) for $U(x)=0$. Then the transfer matrix $\hat{T}$ connecting these solutions

$$\begin{pmatrix} \psi_+(z_0) \\ \psi_-(z_0) \end{pmatrix} = \hat{T} \begin{pmatrix} \psi_+(0) \\ \psi_-(0) \end{pmatrix}$$

and fulfilling the current conservation law

$$|\psi_+(z)|^2 - |\psi_-(z)|^2 = \text{const} \tag{4}$$

is given as follows:

$$\hat{T} = \begin{pmatrix} \left(\dfrac{\gamma+1}{2}\right)^{1/2} \exp(i\varphi_\alpha) & \left(\dfrac{\gamma-1}{2}\right)^{1/2} \exp(i\varphi_\beta) \\ \left(\dfrac{\gamma-1}{2}\right)^{1/2} \exp i(\lambda - \varphi_\beta) & \left(\dfrac{\gamma+1}{2}\right)^{1/2} \exp i(\lambda - \varphi_\alpha) \end{pmatrix} \tag{5}$$

Note that in contrast to the Schrödinger and nontilted ($\beta = 0$) Dirac equations, the transfer matrix in the tilted case depends on four (rather then three) parameters. The point is that in the first two cases there exists a simple procedure that transforms a given solution into a linearly independent one. This transformation is given by a simple complex conjugation in the Schrödinger case, whereas for the nontilted Dirac equation it is given by the charge conjugation (complex conjugation combined with a simultaneous transposition of the components). The existence of these operations reveals itself as additive constraints on the $\hat{T}$-matrix elements, i.e., there are only three independent parameters left in these cases. There is, however, no such operation for the tilted Dirac equation.

The point interaction potentials, which can be naturally thought of as the short-range limit of the potentials in (1), also conserve the current (the corresponding operator is self-adjoint; see the Appendix). The $\hat{T}$-matrix which describes the evolution of the vector

$$\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$$

when passing through the scatterer is obviously parametrized in the same way. In order to distinguish the $\hat{T}$-matrix corresponding to one point scatterer from the $\hat{T}$-matrix describing the whole segment $(0, z_0)$, we use an index 1 labeling all its elements.

The particle transmission through the disordered segment $(0, z_0)$ is described by the corresponding $\hat{T}$-matrix, i.e., by its elements γ, φ_α, φ_β, and λ. The parameter γ is simply connected with the transmission coefficient $T = |T_{11}|^{-2}$,

$$\gamma = 2/T - 1$$

Therefore, in order to find the probability density $w(\gamma \mid z_0)$ of $\gamma(z_0)$, it is enough to integrate the probability density

$$W(\hat{T} \mid z) \equiv w(\gamma, \varphi_\alpha, \varphi_\beta, \lambda \mid z_0)$$

over the remaining variables φ_α, φ_β, and λ:

$$w(\gamma \mid z) = \int w(\hat{T} \mid z)\, d\varphi_\alpha\, d\varphi_\beta\, d\lambda$$

Taking into account the current conservation law (4), it is natural to parametrize the vector $\binom{\psi_+}{\psi_-}$ as follows:

$$\psi_+ = \left(\frac{I-J}{2}\right)^{1/2} e^{i(\kappa+\varphi)}, \qquad \psi_- = \left(\frac{I+J}{2}\right)^{1/2} e^{i(\kappa-\varphi)}$$

Here $I(z) = |\psi_+(z)|^2 + |\psi_-(z)|^2$ is the wave intensity and $-J(z) = |\psi_+(z)|^2 - |\psi_-(z)|^2 = \text{const}$ is the conserved current. The dynamics of the variables I, φ is separated and is described by the following equations [cf. Eq. (7) in ref. 4]:

$$\begin{aligned} &I(z_0) = \gamma I_0 + (\gamma^2 - 1)^{1/2} (I_0^2 - J^2)^{1/2} \cos(\psi); \qquad \psi = 2\varphi_0 + \varphi_\alpha - \varphi_\beta \\ &e^{2i\varphi(z_0)}[I^2(z_0) - J^2]^{1/2} \\ &\qquad = e^{i(\varphi_\alpha + \varphi_\beta - \lambda)}[(\gamma^2-1)^{1/2} I_0 + (I_0^2 - J^2)^{1/2} (\gamma \cos\psi + i \sin\psi)] \end{aligned} \tag{6}$$

where $I_0 = I(0)$ and $\varphi_0 = \varphi(0)$. It can be easily seen that the Jacobian $\mathscr{D}(I, \varphi)/\mathscr{D}(I_0, \varphi_0)$ equals 1 and hence the phase volume $dI\, d\varphi$ is conserved under the transformation (6).

Let us now introduce the probability density of the pair I, φ at point z under a fixed current J:

$$W_J(\Gamma | z)\, d\Gamma; \qquad \Gamma = \{I, \varphi\}, \qquad d\Gamma = dI\, d\varphi$$

Then W_J solves the integral equation

$$\begin{gathered} W_J(\Gamma | z) = \int \delta(\Gamma - \hat{T}\Gamma_0)\, w(\hat{T} | z)\, W_J(\Gamma_0 | 0)\, dT\, d\Gamma_0 \\ d\hat{T} = d\gamma\, d\varphi_\alpha\, d\varphi_\beta\, d\lambda \end{gathered} \tag{7}$$

Here $\hat{T}$ is an operator defined by the system (6) with $z=z_0$ and with the parameters γ, φ_α, φ_β, λ corresponding to the $\hat{T}$-matrix of the segment $(0, z_0)$.

Let the disorder be caused by point impurities which are independently and uniformly distributed over the segment $(0, z_0)$ with a mean density n. Then, taking into account that the segment $(z, z+dz)$ does not contain an impurity with the probability $1-n\,dz$ and does contains an impurity with the probability $n\,dz$, it is easy to find that the probability density $W_J(\Gamma\,|\,z)$ satisfies an equation which follows from (7) (see refs. 1 and 4 for details)

$$\frac{\partial W}{\partial z}+\frac{p}{2}\frac{\partial W}{\partial \varphi}+n(W-\tilde{W})=0$$

where

$$p=p_+(E)-p_-(E), \qquad \tilde{W}=W(I_0(I, \varphi)\,|\,z) \tag{8}$$

In the high-energy limit $p\to\infty$ we can find the solution of this equation as a series in powers of p^{-1}. The zeroth-order approximation is φ independent, i.e.,

$$W_J^{(0)}=W_J^{(0)}(I\,|\,z) \tag{9}$$

and a nontrivial equation for it can be obtained from the condition of the existence of 2π-periodic solutions of first-order approximation equations. It has the form

$$\frac{1}{n}\frac{\partial W_J^{(0)}(I\,|\,z)}{\partial z}=\frac{1}{2\pi}\int_0^{2\pi} W_J^{(0)}(I_0(I, \varphi)\,|\,z)\,d\varphi-W_J^{(0)}(I\,|\,z) \tag{10}$$

On the other hand, using (6) and (7) and an obvious condition $W_1^{(0)}(\gamma\,|\,0)=\delta(\gamma-1)$, it is simple to see that in the same approximation the following equation holds:

$$w^{(0)}(\gamma\,|\,z)=W_1^{(0)}(\gamma\,|\,z)$$

Taking $J=1$ in (10), we can see that the sought probability density $w^{(0)}(\gamma\,|\,z)$ solves the integral equation

$$\frac{1}{n}\frac{\partial w^{(0)}(\gamma\,|\,z)}{\partial z}$$
$$=\frac{1}{2\pi}\int_0^{2\pi} w^{(0)}\{\gamma\gamma_1-[(\gamma^2-1)(\gamma_1^2-1)]^{1/2}\cos 2\varphi\,|\,z\}\,d\varphi-w^{(0)}(\gamma\,|\,z) \tag{11}$$

Here all the arguments of the unknown function are greater than one and it is therefore natural (as in the Schrödinger case[4]) to seek the solution in the form of an integral over the cone functions[5]

$$w^{(0)}(\gamma \mid z) = \int_0^\infty \tilde{w}^{(0)}(t \mid z)\, P_{-1/2+it}(\gamma)\, dt$$

Inserting this representation into (11) and using the addition theorem for the Legendre functions, we find

$$\frac{1}{n} \frac{\partial \tilde{w}^{(0)}(t \mid z)}{\partial z} = [P_{-1/2+it}(\gamma_1) - 1]\, \tilde{w}^{(0)}(t \mid z)$$

and finally

$$w^{(0)}(\gamma \mid z) = \int_0^\infty P_{-1/2+it}(\gamma)\, t \,\mathrm{th}(\pi t) \times \exp\{nz[P_{-1/2+it}(\gamma_1) - 1]\}\, dt$$

This implies, in particular, that the inverse localization length, which is defined as the (self-averaged for $z \to \infty$) decrement of the transmission coefficient on a given realization

$$l^{-1} = -\lim_{z \to \infty} z^{-1} \ln T(z) = -\lim_{z \to \infty} z^{-1} \langle \ln T(z) \rangle$$

can be expressed in the high-energy limit using the density of scatterers and the transmission coefficient for one of them only,

$$l^{-1} = -n \ln T_1$$

This formula coincides with the respective formula in the Schrödinger case and is the same as in the independent-scatterers approximation.[1] Thus, the difference between the Schrödinger and Dirac cases is determined completely by the form of the energy dependence of the parameter γ which describes the scattering on a single point scatterer.

3. THE POINT SCATTERERS FOR THE DIRAC EQUATION

The general form of the $\hat{T}$-matrix of the point scatterer in the weakly tilted Dirac equation is given by the formula (5). This is in full agreement with the existence of a four-parameter family of self-adjoint extensions of the Dirac operator which is the natural candidate for the point interaction Hamiltonians.[6]

In order to obtain a concrete dependence of the $\hat{T}$-matrix elements on the energy, we consider first Eq. (1) with a square well potential

$$\beta U(x)=\begin{cases} U_0, & |x|<a \\ 0, & |x|>a \end{cases} \tag{12}$$

Taking the limit $U_0 \to \infty$, $a \to 0$, $2aU_0 \to k_0 = \text{const}$, the matrix $\hat{T}$ of (5) corresponding to the potential (12) becomes equal to

$$\hat{T}=e^{-2i\alpha_-}\left\{I+(1-e^{2ik_0/v(1-\beta^2)})\begin{pmatrix} \text{sh}^2\,\theta_0, & \text{sh}\,\theta_0\,\text{ch}\,\theta_0 \\ -\text{sh}\,\theta_0\,\text{ch}\,\theta_0, & -\text{ch}^2\,\theta_0 \end{pmatrix}\right\} \tag{13}$$

where

$$\alpha_\pm=\frac{k_0}{2v(1\pm\beta)}, \qquad \text{cth}\,2\theta_0=\frac{E}{E_0}, \qquad E_0=\Delta(1-\beta^2)^{1/2}$$

The single-scatterer reflection coefficient $R_1=1-|\hat{T}_{11}|^{-2}$ is given by

$$R_1=\left[1+\left(\text{sh}^2\,2\theta_0\,\sin^2\frac{k_0}{v(1-\beta^2)}\right)^{-1}\right]^{-1} \tag{14}$$

and we get for the localization length

$$l^{-1}=n\ln\left(1+\text{sh}^2\,2\theta_0\,\sin^2\frac{k_0}{v(1-\beta^2)}\right) \tag{15}$$

Using the natural definition of the integral

$$\int_{-\varepsilon}^{\varepsilon}\delta(x)\,\psi(x)\,dx=\tfrac{1}{2}[\psi(+0)+\psi(-0)]$$

the point perturbation described by the $\hat{T}$-matrix (13) can be written down using the term

$$k_0\delta(x)\,P\psi \tag{16}$$

where P is a diagonal matrix of the form

$$P=\begin{pmatrix} (\text{tg}\,\alpha_-)/\alpha_-, & 0 \\ 0, & (\text{tg}\,\alpha_+)/\alpha_+ \end{pmatrix} \tag{17}$$

In the case of a weak scattering ($\alpha_\pm \ll 1$), $P \to I$ and

$$R=\frac{k_0}{v^2(1-\beta^2)^2}\,\text{sh}^2\,2\theta_0$$

In the limit case $n \to \infty$, $k_0 \to 0$, and $nk_0^2 \to 2d$ we get from (15) an expression for the localization length which corresponds to a white noise potential and which was previously published in refs. 2 and 3:

$$l^{-1} = \frac{2d}{v^2(E-E_0)^2} \frac{E_0^2}{(1-\beta^2)^2} \tag{18}$$

From (13) and (14) it follows that in the case

$$\frac{k_0}{v(1-\beta^2)} = n\pi$$

the point scatterer becomes reflectionless ($R = 0$). The respective $\hat{T}$-matrix is equal to $\exp(-2i\alpha_-)$I, while the potential (12) for $\alpha_\pm \neq (2n+1)\pi/2$ is described by (16) and (17). Let us stress that this fact is a property of the tilted Dirac equation. In the untilted case, $\beta = 0$, the reflectionless potential corresponds to a T-matrix which is equal to $\pm I$, while the potential itself equals either zero or infinity, respectively. The inverse localization length (15) is also equal to zero. Consequently, we can construct a random potential which is made from the reflectionless scatterers and for which even in a typical "localized" weakly tilted case all the states will be delocalized. The state structure is extremely simple: only the phase of the wavefunction changes with the coordinate, being a random, uniformly distributed quantity from the interval $(0, 2\pi)$ at each point x. Let us note that the mechanism of the above-described reflectionless potential is completely different from the mechanism of the so-called Bargmann potentials.[7]

The structure of the self-adjoint extensions for the Dirac equation was previously investigated.[6] In particular, two one-parameter families were distinguished which are in some sense analogous to the point scatterers of the type $\delta(x)$ and $\delta'(x)$ in the Schrödinger equation. In the case of a weakly tilted Dirac equation the $\hat{T}$-matrices corresponding to these scatterers have (in the basis ψ_+, ψ_-) the form

$$\hat{T} = I + \frac{i\alpha}{2} e^{\pm 2\theta_0} \begin{pmatrix} 1, & \pm 1 \\ \mp 1, & -1 \end{pmatrix} \tag{19}$$

Here the upper and lower signs correspond to the scatterer of the type $\delta(x)$ and $\delta'(x)$, respectively. Their Schrödinger analogues are (in the same basis) the following:

$$\hat{T} = I + \frac{i}{2} \left(\frac{k_0}{k}\right)^{\pm 1} \begin{pmatrix} 1, & \pm 1 \\ \pm 1, & -1 \end{pmatrix} \tag{20}$$

Comparing now (19) with (20) and taking into account (13), we can see that in both cases the perturbations of the type $\delta(x)$ become infinitely strong for $E \to E_0$ ($k \to 0$), while the perturbations of the type $\delta'(x)$ tend to zero. In the high-energy limit, however, the behavior of these perturbations in the Schrödinger and tilted Dirac equations becomes quite different. In the Schrödinger case the perturbation of the δ type becomes infinitely weak, while the δ' type becomes infinitely strong. On the other hand, in the tilted Dirac equation both the perturbations δ and δ' tend to a finite limit as $E \to \infty$ which is equal to

$$\hat{T}_{\infty} = I + \frac{i}{2} v k_0 \begin{pmatrix} 1, & \pm 1 \\ \mp 1, & -1 \end{pmatrix}$$

One can introduce the δ-like potential directly also for the tilted Dirac equation. The wavefunctions become discontinuous in the points $x = x_j$ where the scatterers are localized and the corresponding integral is defined as follows:

$$\int_{x_j - 0}^{x_j + 0} \delta(x - x_j) \, |\psi(x)\rangle \, dx = \tfrac{1}{2}[|\psi(x_j + 0)\rangle + |\psi(x_j - 0)\rangle]$$

Equation (1), which describes the system with scatterers of δ type, has the form

$$-iv(\sigma_z - \beta) \frac{d\psi}{dx} + \Delta \sigma_x \psi - v \sum_j k_0 \delta(x - x_j) \, P_\beta \psi = E\psi \tag{21}$$

where the projection P_β is given by

$$P_\beta = \begin{pmatrix} \frac{1}{2}(1-\beta), & \frac{1}{2}(1-\beta^2)^{1/2} \\ \frac{1}{2}(1-\beta^2)^{1/2}, & \frac{1}{2}(1+\beta) \end{pmatrix}$$

For the reflection coefficient corresponding to one point scatterer of the δ type, we find

$$R_1 = \frac{k_0^2}{4v^2} \frac{E + E_0}{E - E_0} \left(\frac{E + E_0}{2} \right)^2$$

Hence, for $k_0 \ll V$ the inverse localization length is equal to

$$l^{-1} = \frac{n k_0^2}{4V^2(E^2 - E_0^2)}$$

and in limit $n \to \infty$, $k_0 \to 0$, $n k_0^2 \to 2d$, it is considerably different from the expression (18). This means that in the Dirac case the point potential in

(21) is analogous to the δ function in the Schrödinger equation in the sense that it gives the same T-matrix in the basis ψ_+, ψ_-, and the point potential (16) obtained as the short-range limit of the square-well potential (12) and generating the white-noise potential corresponds to two different families of self-adjoint extensions. In the Schrödinger case, however, these families coincide.

4. STATISTICAL PROPERTIES OF THE TRANSFORMATION COEFFICIENT IN THE STRONGLY TILTED CASE

In the strongly tilted case both the free solutions propagate in one direction only. The solutions of Eq. (1) on the left and on the right side of the disordered segment $(0, z_0)$ respectively are connected by the transfer matrix $\hat{T}$,

$$\begin{pmatrix} \psi_+(0) \\ \psi_-(0) \end{pmatrix} = \hat{T} \begin{pmatrix} \psi_+(z_0) \\ \psi_-(z_0) \end{pmatrix}$$

which has the form

$$\hat{T} = \begin{pmatrix} \left(\dfrac{1+\gamma}{2}\right)^{1/2} \exp(i\varphi_\alpha) & \left(\dfrac{1-\gamma}{2}\right)^{1/2} \exp(i\varphi_\beta) \\ -\left(\dfrac{1-\gamma}{2}\right)^{1/2} \exp[i(\lambda - \varphi_\beta)] & \left(\dfrac{1+\gamma}{2}\right)^{1/2} \exp[i(\lambda - \varphi_\alpha)] \end{pmatrix} \tag{22}$$

In such a way a wave of the first type which is incident at the right end of the disordered segment $[\psi_-(z_0) = 0]$ is partially transformed into a wave of the second type. The difference of the squared moduli of the free solutions on the right is proportional to the quantity γ. The transformation is absent for $\gamma = 1$, while for $\gamma = -1$ the solutions of the first type are completely transformed into solutions of the second type. It is therefore natural to call γ the disbalance coefficient of these transformation.

Analogously as in Section 2, we introduce a probability density $w(\gamma, \varphi_\alpha, \varphi_\beta, \lambda \mid z)$ which, being integrated over the variables φ_α, φ_β, λ, gives the probability density of the disbalance coefficient. Since in the strongly tilted case the conserved quantity is given by the intensity $|\psi_+(z)|^2 + |\psi_-(z)|^2 = I = \text{const}$ and not by the current $J(z) = |\psi_+(z)|^2 - |\psi_-(z)|^2$, it is natural to parametrize the vector (ψ_+, ψ_-) as

$$\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \begin{pmatrix} \left(\dfrac{I+J}{2}\right)^{1/2} e^{i(\kappa + \varphi)} \\ \left(\dfrac{I-J}{2}\right)^{1/2} e^{i(\kappa - \varphi)} \end{pmatrix}$$

The dynamics of the pair (J, φ) is separated and is determined by

$$\begin{aligned} J(z) &= \gamma J_0 + [(1-\gamma^2)(I^2 - J_0^2)]^{1/2} \cos\psi, \qquad \psi = 2\varphi_0 + \varphi_\alpha - \varphi_\beta \\ [I^2 - J(z)^2]^{1/2} & e^{2i\varphi(z)} \\ &= e^{i(\varphi_\alpha + \varphi_\beta - \lambda)}[-J_0(1-\kappa^2)^{1/2} + (I^2 - J_0^2)(\gamma\cos\psi + i\sin\psi)] \end{aligned} \tag{23}$$

The Jacobian $\mathscr{D}(J, \varphi)/\mathscr{D}(J_0, \varphi_0)$ is again equal to one and the phase space measure is therefore conserved.

Let us introduce further the probability density of the pair (J, φ) at a point $y = z_0 - z$ which corresponds to a fixed "intensity" I,

$$W_I(\Gamma \mid y)\, d\Gamma, \qquad \Gamma = \{J \cdot \varphi\}, \qquad d\Gamma = dJ\, d\varphi$$

Then W solves an integral equation

$$W_I(\Gamma \mid y) = \int \delta(\Gamma - \hat{T}\Gamma_0)\, w(\hat{T} \mid y)\, W_I(\Gamma_0 \mid 0)\, d\hat{T}\, d\Gamma_0 \tag{24}$$

where $\hat{T}$ is an operator given by the system (23).

If the disorder is caused by independent uniformly distributed scatterers localized on the segment $(0, z_0)$ with a density n, then the probability density $W_I(\Gamma \mid y)$ again solves Eq. (8), where now

$$p = p_- - p_+, \qquad \tilde{W}_I = W_I(J_0(J, \varphi) \mid y)$$

The equations and solutions of the zeroth-order approximation in the parameter $1/p$ are obtained in the high-energy limit from (9) and (10) changing $J \leftrightarrow I$ and $z \leftrightarrow y$, respectively, and taking $J_0(J, \varphi)$ from (23). Moreover, we get from (23), (24), and the condition

$$W_I^{(0)}(J \mid 0) = \delta(J - 1)$$

that in the zeroth-order approximation we have

$$w^{(0)}(\gamma \mid y) = W_1^{(0)}(\gamma \mid y)$$

Therefore the integral equation for $w^{(0)}(\gamma \mid z)$ acquires the form

$$\begin{aligned} \frac{\partial w^{(0)}(\gamma \mid y)}{\partial y} &= \frac{n}{2\pi} \int_0^{2\pi} w^{(0)}\{\gamma\gamma_1 - [(1-\gamma^2)(1-\gamma_1^2)]^{1/2} \cos 2\varphi \mid y\}\, d\varphi \\ &\quad - n w^{(0)}(\gamma \mid z) \end{aligned} \tag{25}$$

All the arguments of the unknown function are less than one. It is therefore natural to seek the solution as a series in Legendre polynomials:

$$w^{(0)}(\gamma \mid y) = \sum_{k=0}^{\infty} w_k^{(0)}(z)\, P_k(\gamma)$$

Inserting this series into (25) and using the addition formula for the Legendre polynomials, we find

$$\frac{1}{n}\frac{\partial w_k^{(0)}}{\partial z} = [P_k(\gamma_1) - 1]\, w_k^{(0)}$$

and finally

$$w^{(0)}(\gamma \mid z) = \sum_{k=0}^{\infty} \frac{2k+2}{2} P_k(\gamma)\, e^{nz[P_k(\gamma_1)-1]} \tag{26}$$

From this expression it follows that the mean value of the disbalance coefficient is exponentially small for large z,

$$\langle \gamma \rangle = e^{-2z/l}, \qquad l = \frac{2}{n[1 - P_1(\gamma_1)]}$$

It is therefore natural to call l the mixing length. In the weak scattering case, $\gamma_1 = 1 - 2R_1$, $R_1 \ll 1$, the mixing length is equal to

$$l = (nR_1)^{-1}$$

and the probability density (26) is transformed into

$$w^{(0)}(\gamma \mid z) = \sum_{k=0}^{\infty} \frac{2k+1}{2} P_k(\gamma)\, e^{-k(k+1)z/l}$$

Approximating the point potential by a square well potential with a shrinking support, we find the explicit form of the T-matrix

$$\hat{T} = e^{-2i\alpha_-}\left\{ I + \left[\exp\left(\frac{2ik_0}{v(\beta^2-1)}\right) - 1\right]\begin{pmatrix} \cos^2\theta_0 & \sin\theta_0\cos\theta_0 \\ \sin\theta_0\cos\theta_0 & \sin^2\theta_0 \end{pmatrix}\right\}$$

where $\alpha_\pm = k_0/2v(\beta \pm 1)$ and $\operatorname{ctg} 2\theta_0 = -E/\Delta(\beta^2-1)^{1/2}$.

For the transformation coefficient we get

$$R_1 = |T_{12}|^2 = \sin^2 2\theta_0 \sin^2 \frac{k_0}{v(\beta^2-1)} \tag{27}$$

which yields in the weak transformation limit an expression for the mixing length

$$l^{-1} = \frac{nk_0^2}{2} \frac{\sin^2 2\theta_0}{2}$$

coinciding with those obtained originally in refs. 2 and 3. For

$$\frac{k_0}{v(\beta^2 - 1)} = n\pi$$

a transformationless transition, $R_1 = 0$ in (27), occurs through the particular point scatterer.

Let us note in conclusion that for a sound signal which is a sequence of extremely short (point) pulses, just as for a white noise signal,[2] the rate of nonlinear absorption by a superconductor is significantly higher than that of a periodic signal of finite length and is practically independent of the signal root mean square amplitude (see the respective results for the white noise case in ref. 2).

APPENDIX

Here we consider mathematical questions concerning the point interaction in the tilted Dirac equation. Namely, we construct the self-adjoint operators (Hamiltonians) which correspond to the four-parameter family of $\hat{T}$ matrices (5) and show that they represent short-range limits of the operators corresponding to (1) with local potentials U. We hope that these results, the foundation of our considerations in Sections 3 and 4, are of interest themselves.

A1. Construction of the Hamiltonians

Let us start with a free tilted Dirac operator which is defined on the Hilbet space $\mathscr{H} = L^2(\mathbb{R}) \otimes \mathbb{C}^2$ by the differential expression

$$H = -iv(\sigma_z - \beta) \frac{d}{dx} + \Delta\sigma_x$$

It is simple to prove that H with $D(H) = \{f \in \mathscr{H};\ f \in AC(\mathbb{R}) \otimes \mathbb{C}^2, Hf \in \mathscr{H}\}$ is a self-adjoint operator for all $\beta \in \mathbb{R}$. In order to construct the Hamiltonian leading to the T-matrices (5), we follow the standard procedure when dealing with potentials supported on sets with a measure zero.[7]

The first step is to remove the interaction point by defining a restricted operator H_0

$$H_0 = H \mid C_0^\infty(\mathbb{R}\backslash\{0\}) \otimes \mathbb{C}^2$$

The operator H_0 is symmetric and for $\beta \neq \pm 1$ has deficiency indices equal to (2, 2). The second step is to construct all self-adjoint extensions of H_0. Because the deficiency indices of H_0 equal (2, 2), we have for $\beta \neq \pm 1$ a four-parameter family of such extensions. It is not difficult to show that there is a one-to-one correspondence between these Hamiltonians and the transfer matrices (5). In order to illustrate it, we restrict ourselves to the one-parameter subfamily (19) only. As already mentioned, this family, in the case of the δ-type scatterers, is described formally by Eq. (21). Hence, in order to find the respective family, it suffices to find a self-adjoint realization of the heuristic expression

$$h_\lambda = -iv(\sigma_z - \beta)\frac{d}{dx} + \Delta\sigma_x - v\lambda\,\delta(x)P_\beta$$

Theorem 1. The self-adjoint realization of the operator h_λ is given for $\beta \neq \pm 1$ by

$$H_\lambda = -iv(\sigma_z - \beta)\frac{d}{dx} + \Delta\sigma_x$$

$$D(H_\lambda) = \Big\{ f = (f_1, f_2) \in \mathscr{H},\, f_i \in AC(\mathbb{R}\backslash\{0\}),\, i = 1, 2;$$

$$z_1 f_1(0_-) + \bar{z}_1 f_1(0_+) = -\frac{\lambda}{2}(1-\beta^2)^{1/2}\,[f_2(0_-) + f_2(0_+)]$$

$$z_2 f_2(0_-) + \bar{z}_2 f_2(0_+) = -\frac{\lambda}{2}(1-\beta^2)^{1/2}\,[f_1(0_-) + f_1(0_+)]$$

$$\text{with } z_1 = \left(\frac{\lambda}{2} - iv\right)(1-\beta),\; z_2 = \left(\frac{\lambda}{2} + iv\right)(1+\beta)\Big\}$$

Proof. Let us first prove that the operator H_λ defined by the indicated boundary conditions at the origin actually represents a self-adjoint extension of H_0. We know from the general theory[8,9] that any self-adjoint extension of H_0 is determined by two independent boundary conditions at 0. It is therefore enough to show that the used boundary conditions nullify the boundary form

$$b(f, g) = (f, H_0^* g) - (g, H_0^* f)$$

[here $(\cdot, \cdot)$ denotes the scalar product in $\mathscr{H}$]. A simple calculation leads to

$$b(f, f) = iv(1-\beta)[|f_1(0_+)|^2 - |f_1(0_-)|^2] + iv(1+\beta)[|f_2(0_-)|^2 - |f_2(0_+)|^2]$$

and the rest is verified by inserting the boundary conditions directly into $b(\cdot, \cdot)$. The operator H_λ is hence self-adjoint. In order to show that it represents the realization of the heuristic operator h_λ, we use the formula

$$\int_{x_j-0}^{x_j+0} \delta(x-x_j)\, |\psi(x)\rangle\, dx = \tfrac{1}{2}[|\psi(x_j+0)\rangle + |\psi(x_j-0)\rangle]$$

and insert it into the equation

$$-iv(\sigma_z - \beta)\frac{d\psi}{dx} + \Delta\sigma_x\psi - v\lambda\,\delta(x)\,P_\beta\psi = E\psi$$

An integration by parts leads then to the boundary conditions which determine $D(H_\lambda)$.

A2. The Short-Range Approximation

We show now that the point interaction Hamiltonians which we obtained by applying the von Neumann theory represent a short-range limit of Hamiltonians with local potential. We restrict ourselves to a particular one-parameter family of Hamiltonians, which corresponds to the class of heuristic operators

$$h_\alpha = -i(\sigma_z - \beta)\frac{d}{dx} + \Delta\sigma_x + \alpha\begin{pmatrix} 1; & -1 \\ -1; & 1\end{pmatrix}\delta(x)$$

We show that the h_α are short-range limits of Hamiltonians of the type

$$H_\varepsilon = -i(\sigma_z - \beta)\frac{d}{dx} + \Delta\sigma_x + \frac{1}{\varepsilon}\begin{pmatrix} 1; & -1 \\ -1; & 1\end{pmatrix} V\left(\frac{x}{\varepsilon}\right), \qquad V \in C_0^\infty(\mathbb{R})$$

Let us now pass to the precise mathematical formulations of the above statements. We introduce a one-parameter family of self-adjoint operators H_α which represents a self-adjoint realization of the heuristic operators h_α:

$$H_\alpha = -i(\sigma_z - \beta)\frac{d}{dx} + \Delta\sigma_x$$

$$D(H_\alpha) = \left\{ f = (f_1, f_2) \in \mathscr{H}, f_i \in AC(\mathbb{R}\backslash\{0\}); i = 1, 2; \right.$$

$$z_1 f_1(0_-) + \bar{z}_1 f_1(0_+) = \frac{\alpha}{2}\,[f_2(0_-) + f_2(0_+)]$$

$$z_2 f_2(0_-) + \bar{z}_2 f_2(0_+) = \frac{\alpha}{2}\,[f_1(0_-) + f_1(0_+)]$$

$$\left. \text{with } z_1 = \frac{\alpha}{2} + i(\beta - 1);\ z_2 = \frac{\alpha}{2} + i(\beta + 1) \right\}$$

(In order to show that H_α actually represents the self-adjoint realization of h_α, one has to follow step by step the proof of Theorem 1.)

Theorem 2.

$$H_\alpha = \underset{\varepsilon \to 0}{\text{N.R.-lim}}\, H_\varepsilon$$

with α given by

$$\alpha = -\frac{1}{2(1+\beta)}\,(u, (1+T)^{-1}\, v)_{L^2(\mathbb{R})} \tag{A.1}$$

where $u(x) = |V(x)|^{1/2}$, $v(x) = |V(x)|^{1/2}\,\mathrm{sgn}(V(x))$, and T is a Hilbert–Schmidt operator on $L^2(\mathbb{R})$ with a kernel

$$T(x, y) = \frac{i\beta}{2(1-\beta^2)}\, v(x)\,\mathrm{sgn}(x-y)\, u(y)$$

N.R.-lim means the limit in the norm-resolvent topology and $(\cdot\,;\cdot)_{L^2(\mathbb{R})}$ denotes the scalar product in $L^2(\mathbb{R})$.

We split the proof into several lemmas.

Lemma 1. The operators H_α and H_ε are unitary equivalent to

$$\tilde{H}_\alpha = -i(\sigma_x - \beta)\frac{d}{dx} - \Delta\sigma_z$$

$$D(\tilde{H}_\alpha) = \left\{ f = (f_1, f_2) \in \mathscr{H}, f_i \in AC(\mathbb{R}\backslash\{0\}); i = 1, 2; \right.$$

$$f_1(0_-) - f_1(0_+) = \beta[f_2(0_-) - f_2(0_+)]$$

$$\left. f_2(0_-) - \bar{z}_2 f_2(0_+) = \frac{i\alpha}{2}\,[f_1(0_-) + f_1(0_+)] \right\}$$

and

$$\tilde{H}_\varepsilon = -i(\sigma_y - \beta)\frac{d}{dx} - \Delta\sigma_z + \frac{2}{\varepsilon}\begin{pmatrix} 1; & 0 \\ 0; & 0 \end{pmatrix} V\left(\frac{x}{\varepsilon}\right)$$

respectively. The unitary mapping is in both cases given by a constant matrix

$$U = (1/2)^{1/2}\begin{pmatrix} 1; & -1 \\ 1; & 1 \end{pmatrix}$$

Proof. Direct verification.

From this lemma it follows that in order to prove Theorem 2 it is enough to prove that

$$\tilde{H}_\alpha = \text{N.R.-}\lim_{\varepsilon \to 0} \tilde{H}_\varepsilon$$

with α given by the expression (A.1). From now we will work only with the operators $\tilde{H}_\alpha$, $\tilde{H}_\varepsilon$,.... We drop the tilde in all expressions.

Lemma 2. The family H_α, $\alpha \in \mathbb{R} \cup \{\infty\}$, exhausts all self-adjoint extensions of the operator

$$H_0 = -i(\sigma_x - \beta)\frac{d}{dx} - \Delta\sigma_z$$

$$D(H_0) = \{f = (f_1, f_2) \in \mathscr{H}, f_i \in AC(\mathbb{R});\ i = 1, 2 \text{ and } f_1(0) = 0\}$$

Proof. The operator H_0 has deficiency indices equal to $(1, 1)$. This can be easily seen when we evaluate the corresponding adjoint operator H_0^*, which is determined by

$$D(H_0^*) = \{f = (f_1, f_2) \in \mathscr{H}, f_i \in AC(\mathbb{R}\backslash\{0\});\ i = 1, 2 \text{ and}$$
$$f_1(0_-) - f_1(0_+) = \beta[f_2(0_-) - f_2(0_+)]\}$$

Hence all the functions from $D(H_0^*)$ are already constrained by one boundary condition at 0 [note that $D(H_0^*) \neq D(H_0)$ and the deficiency indices are therefore not zero]. Consequently, H_0 has exactly one parameter family of self-adjoint extensions. This family must, however, coincide with H_α because $D(H_\alpha) \subset D(H_0^*)$ for all $\alpha \in \mathbb{R}$.

Knowing that the H_α represent the self-adjoint extension of H_0, we can apply the Krein formula[8] and obtain the corresponding resolvent. We restrict ourselves to the case $\beta < 1$ and we calculate the resolvent only for the spectral parameter equal to zero.

Lemma 3.

$$H_\alpha^{-1} = H^{-1} + \frac{\alpha}{2(1-\beta^2)[1-\beta-\alpha(1-\beta^2)^{1/2}]} |f_1\rangle\langle f_2|$$

where $|f_1\rangle\langle f_2|$ is a rank-one operator,

$$|f_1\rangle\langle f_2|: \quad f \to f_1(f_2; f)$$

with

$$f_1(x) = \begin{pmatrix} \beta\, \mathrm{sgn}(x) - (\beta^2-1)^{1/2} \\ \mathrm{sgn}(x) \end{pmatrix} \exp\left(- \frac{\Delta\, |x|}{(1-\beta^2)^{1/2}} \right)$$

$$f_2(x) = \begin{pmatrix} \beta\, \mathrm{sgn}(x) + (\beta^2-1)^{1/2} \\ \mathrm{sgn}(x) \end{pmatrix} \exp\left(- \frac{\Delta\, |x|}{(1-\beta^2)^{1/2}} \right)$$

H^{-1} is an integral operator with the kernel

$$H^{-1}(x, y) = \frac{i}{2(1-\beta^2)} \begin{pmatrix} \beta\, \mathrm{sgn}(x-y) - (\beta^2-1)^{1/2}; & \mathrm{sgn}(x-y) \\ \mathrm{sgn}(x-y); & \beta\, \mathrm{sgn}(x-y) + (\beta^2-1)^{1/2} \end{pmatrix} \times \exp\left(- \frac{\Delta\, |x-y|}{(1-\beta^2)^{1/2}} \right)$$

Proof. Direct evaluation shows that for all $\varphi \in \mathscr{H}$

$$H_\alpha^{-1} \varphi \in D(H_\alpha)$$

and

$$H_\alpha H_\alpha^{-1} \varphi = \varphi$$

Proof of Theorem 2. We suppose that $\beta < 1$ and that $V \geq 0$. In order to indicate explicitly the dependence of the resolvent on the parameter Δ, we will write $H^{-1}(\Delta)$, $H_\alpha^{-1}(\Delta)$, and $H_\varepsilon^{-1}(\Delta)$ instead of H^{-1}, H_α^{-1}, and H_ε^{-1}. Introducing the scaling group

$$U_\varepsilon: f(x) \to \varepsilon^{-1/2} f(x/\varepsilon)$$

we find for $H_\varepsilon^{-1}(\Delta)$

$$\begin{aligned} H_\varepsilon^{-1}(\Delta) &= H^{-1}(\Delta) - \frac{1}{\varepsilon} H^{-1}(\Delta)\, U_\varepsilon v \mathbb{P}(1 + v\mathbb{P} H^{-1}(\varepsilon\Delta)\, \mathbb{P} v)^{-1}\, \mathbb{P} v U_\varepsilon^{-1} H^{-1}(\Delta) \\ &= H^{-1}(\Delta) - A_\varepsilon (1 + B_\varepsilon)^{-1}\, C_\varepsilon \end{aligned}$$

where

$$v(x)=V(x)^{1/2}, \qquad \mathbb{P}=\begin{pmatrix}1, & 0\\ 0, & 0\end{pmatrix}$$

and A_ε, B_ε, and C_ε are Hilbert–Schmidt operators with kernels

$$\begin{aligned}
A_\varepsilon(x, y) &= H^{-1}(\Delta, x, \varepsilon y)\,\mathbb{P}(y)\\
B_\varepsilon(x, y) &= v(x)\,\mathbb{P}H^{-1}(\varepsilon\Delta, x, y)\,\mathbb{P}v(y)\\
C_\varepsilon(x, y) &= v(x)\,\mathbb{P}H^{-1}(\Delta, \varepsilon x, y)
\end{aligned}$$

[$H^{-1}(\Delta, x, y)$ denotes the kernel of $H^{-1}(\Delta)$; cf. Lemma 3.] Taking now the $\varepsilon\to 0$ limit, we find that the operators A_ε, B_ε, C_ε converge in norm to A, B, C, where A and C are rank-one operators with kernels

$$A(x, y)=\frac{i}{2(1-\beta^2)}\begin{pmatrix}\beta\,\mathrm{sgn}(x)-(\beta^2-1)^{1/2}; & 0\\ \mathrm{sgn}(x); & 0\end{pmatrix}\exp\left(\frac{\Delta\,|x|}{(1-\beta^2)^{1/2}}\right)\mathbb{P}v(y)$$

$$C(x, y)=\frac{iv(x)}{2(1-\beta^2)}\,\mathbb{P}\begin{pmatrix}\beta\,\mathrm{sgn}(x)-(\beta^2-1)^{1/2}; & \mathrm{sgn}(x)\\ 0; & 0\end{pmatrix}\exp\left(-\frac{\Delta\,|x|}{(1-\beta^2)^{1/2}}\right)$$

and the operator B is described by

$$B(x, y)=\frac{i}{2(1-\beta^2)}\begin{pmatrix}\beta v(x)\,\mathrm{sgn}(x-y)\,v(y)-v(x)\,v(y)(\beta^2-1)^{1/2}; & 0\\ 0; & 0\end{pmatrix}$$

Thus

$$\underset{\varepsilon\to 0}{n\text{-lim}}\; H_\varepsilon^{-1}=H^{-1}(\Delta)-A(1+B)^{-1}\,C \tag{A.2}$$

To invert the operator $(1+B)$, we split the operator B into two parts:

$$B=B_1+T$$

with B_1 being a rank-one operator

$$B_1=\frac{1}{2(1-\beta^2)^{1/2}}\begin{pmatrix}v(x)\,v(y), & 0\\ 0, & 0\end{pmatrix}$$

We get then

$$(1+B)^{-1}=[1+(1+T)^{-1}\,B_1]^{-1}\,(1+T)^{-1}$$

(Note that a rank-one operator can be inverted explicitly.) Inserting this formula into and comparing the result with the formula for H_α^{-1}, we obtain the assertion of the theorem.

The coupling constant α is a quite irregular function of the potential V. In order to illustrate this peculiarity, we multiply the potential V by a constant λ. In such a way α becomes a function of λ,

$$\alpha(\lambda) = -\frac{\lambda}{2(1+\beta)}(u, (1+\lambda T)^{-1} v)_{L^2(\mathbb{R})}$$

The function $\alpha(\lambda)$ is, however, not smooth. It has singularities which are localized at points λ_k for which $-1/\lambda_k$ is an eigenvalue of the operator T. On the other hand, T is Hilbert–Schmidt and the behavior of its eigenvalues is very well known.[10] Applying results from ref. 10, we get the following assertion:

Let $N(\Lambda)$ be the number of singularities of the function $\alpha(\lambda)$ which are localized in an interval $[0, \Lambda]$:

$$N(\Lambda) = \#\{\lambda \in [0, \Lambda]; \alpha^{-1}(\lambda) = 0\}$$

Then

$$\lim_{\Lambda \to \infty} \frac{N(\Lambda)}{\Lambda} = \frac{\beta}{2\pi(1-\beta^2)} \int_{\mathbb{R}} V(x)\, dx$$

To illustrate the behavior of $\alpha(\lambda)$ in more detail, we choose $V(x)$ in a square well form

$$V(x) = \begin{cases} 0; & x \notin [0, 1] \\ V_0; & x \in [0, 1] \end{cases}$$

and decompose $\alpha(\lambda)$ with respect to λ,

$$\alpha(\lambda) = -\frac{1}{2(1+\beta)} [\lambda(u, v) - \lambda^2(u, Tv) + \cdots + (-1)^n \lambda^{n+1}(u, T^n v) + \cdots] \tag{A.3}$$

For $(u, T^n v)$ we get

$$(u, T^{2n+1} v) = 0 \qquad \text{for all } n$$

$$(u, T^{2n} v) = (-1)^n V_0^{2n+1} \left[\frac{\beta}{2(1-\beta^2)}\right]^{2n} \times \int_0^1 \int_0^1 \cdots \int_0^1 \operatorname{sgn}(x_1 - x_2) \operatorname{sgn}(x_2 - x_3) \cdots \times \operatorname{sgn}(x_{2n} - x_{2n+1})\, dx_1\, dx_2 \cdots dx_{2n+1}$$

Noting that

$$\int_0^1 \int_0^1 \cdots \int_0^1 \operatorname{sgn}(x_1 - x_2) \operatorname{sgn}(x_2 - x_3) \cdots \operatorname{sgn}(x_{2n} - x_{2n+1})\, dx_1\, dx_2 \cdots dx_{2n+1}$$

$$= \frac{2^{2(n+1)}(2^{2(n+1)} - 1)}{(2n+2)!} B_{2n+2}$$

where B_{2n} are the Bernoulli numbers and inserting this result into (A.3), we get finally

$$\alpha(\lambda) = \frac{1-\beta}{\beta} \operatorname{tg}\left(\frac{\lambda V_0 \beta}{2(1-\beta^2)}\right)$$

which is in good correspondence with the result obtained for the short-range approximation of the transfer matrix (13).

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